Scalar Exchange Forces and Generalized Most Attractive Channel Rule

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ABSTRACT

We discuss the possibility that fermionic condensates arise from the dominance of scalar exchange forces over vector gluon exchange. When a scalar in the adjoint representation is exchanged in the reaction $A + A \rightarrow A + A$, the usual most attractive channel (MAC) rule is reversed in sign, with the consequence that the formation of a condensate with the largest possible Casimir is favored. More generally, when a scalar in a general representation is exchanged in the reaction $A + B \rightarrow B + A$, the group theoretic sum giving the force sign and strength can be expressed in terms of a Racah coefficient for the group in question. We illustrate the formalism in the case of the group $SU(2)$, and give possible applications to $SO(10)$ and E_6 grand unification.

Attempts to construct realistic grand unified models typically involve a complicated Higgs scalar sector [1], and this has led to the suggestion [2] that some of these scalars may be dynamically generated fermionic composites. A particularly appealing scenario in this regard is the "tumbling" hypothesis [3], which suggests that a symmetry breaking hierarchy develops in which scalar composites, formed from the fermions of the theory at each level of symmetry breaking, develop vacuum expectation values that trigger the next stage of symmetry breaking. Applications of the tumbling scenario typically assume that the forces giving rise to condensate formation arise from vector gluon exchange. However, this assumption leads to the problem that since the most attractive channel (MAC) rule [4] for vector exchange favors the formation of composites in small representations, it is not possible to generate the large Higgs representations needed for the construction of realistic models.

Our purpose in this Letter is to explore the possibility that scalar mediated forces may play a significant role in dynamical symmetry breaking, and to develop an analog of the MAC rule in this case. We begin with a review of the MAC rule in the case in which a vector gluon mediates the reaction $A + B \rightarrow A + B$. Since a gauge gluon couples with universal strength to the representations A and B, the couplings g_A and g_B are equal, $g_A = g_B = g$. Thus the relevant static potential is

$$\frac{g^2 \, T_A \cdot T_B}{r} \quad , \tag{1a}$$

with $\cdot$ denoting a sum over the group generators and with $T_{A,B}$ the generator matrices for the respective representations A, B. Using

$$T_A \cdot T_B = \frac{1}{2}[(T_A + T_B)^2 - T_A^2 - T_B^2] \quad , \tag{1b}$$

together with the fact that the summed squares of the generators define the respective

quadratic Casimir operators C_2, the effective force strength becomes

$$\frac{g^2[C_2(A+B) - C_2(A) - C_2(B)]}{2r} \quad . \tag{1c}$$

This gives the MAC rule for vector gluon exchange, which states that the interaction is attractive when the square bracket in Eq. (1c) is negative, and that the MAC is the channel with the smallest Casimir for the composite $A + B$.

Let us consider now the case in which a scalar mediates the reaction $A + B \to A + B$. Since scalar couplings are not universal and can have either sign, in general we have $g_A \neq g_B$, and whether the scalar exchange force is attractive or repulsive depends on dynamical details of the Yukawa couplings. However, there are cases of physical interest in which a dynamics independent statement can be made. The simplest is that in which $A = B$, so that $g_A = g_B = g$, and in which the exchanged scalar is in the adjoint representation. The calculation is then the same as that given above for the vector gluon case [5], except that the Coulomb potential $1/r$ is replaced by the Yukawa potential $-\exp(-\mu r)/r$, with μ the scalar mass and with the change in sign reflecting the fact that the Yukawa force is *attractive*, rather than repulsive, for the case of equal couplings at the two vertices. Thus Eq. (1c) becomes in this case

$$-\frac{g^2[C_2(A+A) - 2C_2(A)]\exp(-\mu r)}{2r} \quad , \tag{2}$$

and the MAC is now the channel with the *largest* Casimir for the composite $C = A + A$.

More generally, let us consider the reaction $A + B \to B + A$ with exchange of a scalar in a general representation S, which is emitted at a vertex where fermion representation B changes to fermion representation A, and then is absorbed at a vertex where fermion representation A changes to fermion representation B. Since the processes at the two vertices

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$B \to A + S$ and $A + S \to B$ are inverse to each other, the Yukawa coupling at one will be the complex conjugate of the Yukawa coupling at the other, and a dynamics independent statement about the sign and magnitude of the scalar mediated force is possible. We will be particularly interested in the dependence of the force on the representation of the composite $C = A + B$.

To carry out this calculation, it is helpful to consider the general S exchange reaction $A + B \to A' + B'$. The Bethe-Salpeter kernel corresponding to this process is

$$K(A'm'_A, B'm'_B | Am_A, Bm_B) \quad , \tag{3a}$$

with $m_{A,B}, m'_{A,B}$ the "magnetic" quantum numbers (the eigenvalues of the generators in the Cartan subalgebra) for the respective representations. The corresponding operator K acting in the Hilbert space of the composites is then

$$
\begin{aligned}
K &= \sum_{m_{A,B}, m'_{A,B}} |A'm'_A\rangle |B'm'_B\rangle K(A'm'_A, B'm'_B | Am_A, Bm_B) \langle Am_A| \langle Bm_B| \\
&= \sum_{Cm_C, C'm'_C} |C'm'_C\rangle K(C'm'_C | Cm_C) \langle Cm_C| \quad ,
\end{aligned}
\tag{3b}
$$

where in the second line we have decomposed the direct product states into irreducible representations, adopting the convention that any indices that enumerate representations appearing more than once are included in the representation labels C and C'. Since we are assuming that the S exchange interaction is group invariant, we must have

$$K(C'm'_C | Cm_C) = K_C \delta_{C,C'} \delta_{m_C, m'_C} \quad , \tag{4a}$$

and so Eq. (3b) takes the form

$$K = \sum_{C, m_C} |Cm_C\rangle K_C \langle Cm_C| \quad . \tag{4b}$$

Hence the eigenvalue K_C which determines the force strength as a function of the representation C [which is the generalization of the expression $g^2[C_2(A+B) - C_2(A) - C_2(B)]$ of Eq. (1c)] is given by

$$K_C = \sum_{m_{A,B}, m'_{A,B}} \langle Cm_C|A'm'_A B'm'_B\rangle K(A'm'_A, B'm'_B|Am_A, Bm_B)\langle Am_A Bm_B|Cm_C\rangle \quad . \quad (5a)$$

To calculate the group theoretic part of the kernel of Eq. (3a), we note that there are two contributions: one in which a vertex for the process $B \to B' + S$ is joined to a vertex for $A + S \to A'$, and one in which a vertex for $A \to A' + S$ is joined to a vertex for $B + S \to B'$. Each vertex, by the Wigner-Eckart theorem, is the product of a Clebsch which carries the magnetic quantum number dependence, times a reduced matrix element which is independent of the magnetic quantum numbers. Taking account of the facts that the interaction Lagrangian associated with the vertex process $B \to B' + S$ is the Hermitian adjoint of that associated with the process $B' + S \to B$, and that the two contributions to the kernel are related by the interchange of the labels A and B, we have

$$K(A'm'_A, B'm'_B|Am_A, Bm_B) = \sum_{m_S} [g_S(BB')^* g_S(A'A)\langle Bm_B|B'm'_B Sm_S\rangle^* \langle A'm'_A|Am_A Sm_S\rangle$$
$$+ g_S(AA')^* g_S(B'B)\langle Am_A|A'm'_A Sm_S\rangle^* \langle B'm'_B|Bm_B Sm_S\rangle] \quad .$$
$$(5b)$$

Self-adjointness of this kernel was assumed in the spectral analysis of Eqs. (3a) - (4b), and this is manifest from the expression in Eq. (5b),

$$K^*(Am_A, Bm_B|A'm'_A, B'm'_B) = K(A'm'_A, B'm'_B|Am_A, Bm_B) \quad . \quad (5c)$$

Substituting Eq. (5b) into Eq. (5a), specializing to the case of interest in which $A' = B$ and

$B' = A$, and interchanging the summation indices m'_A and m'_B, we get the result

$$K_C = \sum_{m_{A,B},m'_{A,B},m_S} \langle Cm_C|Bm'_B Am'_A \rangle$$

$$\times [|g_S(BA)|^2 \langle Bm_B|Am'_A Sm_S \rangle^* \langle Bm'_B|Am_A Sm_S \rangle \qquad (6a)$$

$$+|g_S(AB)|^2 \langle Am_A|Bm'_B Sm_S \rangle^* \langle Am'_A|Bm_B Sm_S \rangle]$$

$$\times \langle Am_A Bm_B|Cm_C \rangle \quad .$$

Changing to the standard notation for the Clebsches,

$$\langle Am_A Bm_B|Cm_C \rangle \equiv (Am_A Bm_B|ABCm_C) \quad , \qquad (6b)$$

adopting a phase convention in which the Clebsches are real, and using the symmetry property

$$(Am_A Bm_B|ABCm_C) = \epsilon(A,B,C)(Bm_B Am_A|BACm_C) \quad , \qquad (6c)$$

with $\epsilon(A,B,C) = \epsilon(B,A,C)$ a phase factor of ± 1, Eq. (6a) takes the form

$$K_C = |g_S(BA)|^2 \sum_{m_{A,B},m'_{A,B},m_S} (BACm_C|Bm'_B Am'_A)(ASBm'_B|Am_A Sm_S)$$

$$\times (ASBm_B|Am'_A Sm_S)^*(Am_A Bm_B|ABCm_C) + A \leftrightarrow B$$

$$= |g_S(BA)|^2 \epsilon(A,S,B) \sum_{m_{A,B},m'_{A,B},m_S} (BACm_C|Bm'_B Am'_A)(ASBm'_B|Am_A Sm_S) \qquad (7)$$

$$\times (Sm_S Am'_A|SABm_B)(Am_A Bm_B|ABCm_C) + A \leftrightarrow B$$

$$= |g_S(BA)|^2 \epsilon(A,S,B)((AS)B,A,C|A,(SA)B,C)$$

$$+|g_S(AB)|^2 \epsilon(B,S,A)((BS)A,B,C|B,(SB)A,C) \quad ,$$

where on the final line we have introduced the standard definition [6, 7] of the recoupling coefficient (the Racah coefficient) for three group representations. Equation (7) is our final result for the case of general representations A, B, S, C of a general Lie group. When the representation S is not self-conjugate, only one of the two reduced matrix elements on the

right hand side of Eq. (7) will be nonzero; when the representation S is self-conjugate, there is no physical distinction between the two contributions to the kernel, and we expect symmetries of the Clebsches to collapse the two terms on the right hand side of Eq. (7) into a single term.

Let us briefly examine some special cases of this formula when the Lie group is $SU(2)$, for which the relevant recoupling coefficients are those given in the standard angular momentum texts. Letting $j_{A,B,C,S}$ be the angular momentum values corresponding to the respective representation labels A, B, C, S, we have

$$\epsilon(A, S, B) = (-1)^{j_A + j_S - j_B} , \quad \epsilon(B, S, A) = (-1)^{j_B + j_S - j_A} , \tag{8a}$$

and

$$((j_A j_S) j_B, j_A, j_C | j_A, (j_S j_A) j_B, j_C) = (2j_B + 1)(-1)^{-(2j_A + j_S + j_C)} \begin{Bmatrix} j_A & j_S & j_B \\ j_A & j_C & j_B \end{Bmatrix} . \tag{8b}$$

Substituting these expressions into Eq. (7) and using the permutation symmetries of the 6-j symbols, we get for $SU(2)$

$$K_C = [(2j_B + 1)|g_S(BA)|^2 + (2j_A + 1)|g_S(AB)|^2](-1)^{-(j_A + j_B + j_C)} \begin{Bmatrix} j_C & j_A & j_B \\ j_S & j_A & j_B \end{Bmatrix} . \tag{8c}$$

We now use Eq. (8c) to examine some special cases of interest. When the exchanged scalar has spin 0, we find from the formula given in Eq. (6.3.2) of [6] that

$$\begin{Bmatrix} j_C & j_A & j_B \\ 0 & j_A & j_B \end{Bmatrix} = \delta_{A,B}(-1)^{j_A + j_B + j_C}[(2j_A + 1)(2j_B + 1)]^{-\frac{1}{2}} , \tag{9a}$$

and so Eq. (8c) vanishes when $A \neq B$, and when $A = B$ gives

$$K_C = 2|g_S(AA)|^2 , \tag{9b}$$

which as one would expect is independent of the composite representation C. When the exchanged scalar has spin 1, which is the adjoint representation for $SU(2)$, we find from Table 5 of [6] that when $j_B = j_A$, we have

$$\begin{Bmatrix} j_C & j_A & j_A \\ 1 & j_A & j_A \end{Bmatrix} = (-1)^{1+2j_A+j_C} \frac{2j_A(j_A + 1) - j_C(j_C + 1)}{j_A(2j_A + 1)(2j_A + 2)} \quad , \tag{10a}$$

which gives

$$K_C = \frac{|g_S(AA)|^2}{j_A(j_A + 1)} [j_C(j_C + 1) - 2j_A(j_A + 1)] \quad , \tag{10b}$$

reproducing the C dependence found from the Casimir analysis of Eq. (2).

The other nonvanishing case with $j_S = 1$ is that with $j_B = j_A + 1$; from Table 5 of [6] we see that in this case K_C is positive and is also monotonically increasing with the composite spin j_C. However, once we go to larger representations S than the adjoint representation, it is *not* always true that K_C monotonically increases with the size of the representation C. For example, for $j_S = 2$ and $j_A = j_B = 3$, the allowed range of j_C is from 0 to 6. For these parameters, we have

$$\begin{aligned}
(-1)^{-(j_A+j_B+j_C)} &\begin{Bmatrix} j_C & j_A & j_B \\ j_S & j_A & j_B \end{Bmatrix} \\
&= (-1)^{-j_C} \begin{Bmatrix} j_C & 3 & 3 \\ 2 & 3 & 3 \end{Bmatrix} \\
&= \frac{360 - 47j_C(j_C + 1) + j_C^2(j_C + 1)^2}{2520} \quad ,
\end{aligned} \tag{11}$$

which for the allowed range of j_C assumes a maximum value of $1/7$ at $j_C = 0$, and takes the smaller value $5/84$ at $j_C = 6$.

We turn now to a discussion of possible applications of these results to symmetry breaking in grand unified theories. We begin with the $SO(10)$ model, with a 16 of chiral fermions. One of these fermions is a singlet under $SU(5)$, and in the "seesaw" mechanism

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of Gell-Mann, Ramond, and Slansky [8], is given a large mass by a Higgs field in the 126 representation of $SO(10)$ that acquires a vacuum expectation value. From the viewpoint of descent from an E_6 unifying group, a 126 of $SO(10)$ is ugly, since the smallest E_6 representation [9] giving rise to it under $E_6 \rightarrow SO(10) \times U(1)$ is the $\overline{351'}$. So it is natural to consider the possibility that the 126 Higgs is a composite. Let us suppose that a Higgs scalar in the adjoint 45 representation of $SO(10)$ is present, which can be obtained by descent from the adjoint 78 of E_6. Since in $SO(10)$ we have $16 \times 45 \supset 16$, exchange of the scalar 45 can mediate the process $16 + 16 \rightarrow 16 + 16$, for which the formula of Eq. (2) applies. This tells us that the MAC is the channel with the largest Casimir appearing in $16 \times 16 = 10_s + 120_a + 126_s$, which is the 126_s. Since the two chiral fermion fields in the 16 anticommute, but have their spinor indices contracted with an antisymmetric two index tensor, the group theoretic part of the composite wave function must be symmetric, a requirement satisfied by the 126_s. We note finally that when this composite 126 obtains an $SU(5)$ singlet vacuum expectation value, this can only involve the components of the 16 which are $SU(5)$ singlets, since none of the $SU(5)$ tensor products 10×10, $10 \times \overline{5}$, or $\overline{5} \times \overline{5}$ contains an $SU(5)$ singlet. Hence the chiral symmetries of the components of the $SO(10)$ 16 that are in the $\overline{5}$ and 10 of $SU(5)$ are preserved.

We give next some possible model building applications of the more general analysis leading to Eq. (7). The first application is again to the generation of a composite 126 of $SO(10)$. Since in $SO(10)$, $210 \times 16 \supset 16$, exchange of a scalar 210 can also mediate the process $16 + 16 \rightarrow 16 + 16$. This is the $A = B$ case of Eq. (7); identifying the MAC here will require computing the $SO(10)$ Clebsch interchange phase and Racah coefficient appearing in Eq. (7). The second application concerns the possibility of generating a 45 or 210 of $SO(10)$, starting

with an E_6 model containing a 27 of scalars, one or more chiral fermion families in the 27, and extra pairs $27 + \overline{27}$ of chiral fermions. Since under $E_6 \supset SO(10) \times U(1)$ the 27 decomposes as $27 = 1(4) + 10(-2) + 16(1)$, and since in $SO(10)$ we have $10 \times \overline{16} \supset 16$, exchange of a $10(-2)$ scalar can mediate the process $16(1) + \overline{16}(-1) \rightarrow \overline{16}(-1) + 16(1)$. This corresponds to Eq. (7) with $A = \overline{16}$, $B = 16$, $S = 10$. The possible composites C are 1, 45, and 210 of $SO(10)$, corresponding to the decomposition $\overline{16} \times 16 = 1 + 45 + 210$; to identify the MAC will again require computation of the relevant $SO(10)$ phase factor and Racah coefficient. Another $SO(10) \times U(1)$ case to which our analysis applies is $10(-2) + 10(2) \rightarrow 10(2) + 10(-2)$, which can be mediated by $1(4)$ exchange, and has 1_s, 45_a, and 54_s as equally attractive composite channels. Our final application, which is an extension of the second, is to the symmetry breaking $E_6 \supset SO(10) \times U(1)$ in an E_6 model with fermion and scalar content as just described. Since in E_6 we have $27 \times 27 \supset \overline{27}$, exchange of a scalar 27 can mediate the process $27 + \overline{27} \rightarrow \overline{27} + 27$, corresponding to Eq. (7) with $A = 27$, $B = \overline{27}$, $S = 27$. The possible composites are the representations 1, 78, 650 appearing in the decomposition of $27 \times \overline{27}$, and determining the MAC in this example will require computation of the E_6 phase factor and Racah coefficient appearing in Eq. (7). Further examples of Eq. (7) in the same model can be obtained by proceeding down either of the symmetry breaking chains $SO(10) \rightarrow SU(5) \times U(1)$ or $SO(10) \rightarrow SU(2) \times SU(2) \times SU(4)$.

Acknowledgments

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